

# Applications of the Quon Algebra: 3-D Harmonic Oscillator and the Rotor Model

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## Abstract

In this work we present a method to build in a systematic way a many-body quon basis state. In particular, we show a closed expression for a given number  $N$  of quons, restricted to the permutational symmetric subspace, which belongs to the whole quonic space. The method is applied to two simple problems: the three-dimensional harmonic oscillator and the rotor model and compared to previous quantum algebra results. The differences obtained and possible future applications are also discussed.

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# 1 Introduction

Quons are particles that violate statistics by a small amount, which is controlled by a single parameter  $q$ [1],[2]. The range of variation of the parameter is between  $-1$  and  $+1$ , and the limits of the interval correspond respectively to fermionic and bosonic statistics. The particular commutation relations obeyed by quons define an algebra (the so called quon algebra), which, for a single degree of freedom, gives results very similar to the ones obtained using deformed (or quantum) algebras[3], once we keep the  $q$  interval as above defined. For more than a single degree of freedom however, there are some important differences between both algebras. One consequence of those differences is that it is possible to define quonic operators that behave as irreducible  $su(2)$  tensors[4]. In other words, it is possible to assume that the quon algebra follows the usual angular momentum coupling rules. This last result opens up the possibility of applying the quon algebra to the study of many-body systems, with some important technical advantages over deformed algebras. However, those very same differences also introduce some complications when we try to build many-body quonic states. The "q-mutation" relation that defines the quon algebra is given by  $[a_i, a_j^\dagger]_q = a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}$ , with the additional condition  $a_i |0\rangle = 0$  and  $|0\rangle$  being the quon vacuum state. Unlike quantum algebras, this relation prevents us from establishing any commutation relation between two creation or two annihilation operators, unless  $q$  is either  $+1$  or  $-1$ . Although no such rule is needed to calculate vacuum matrix elements of polynomials in the  $a$ 's or  $a^\dagger$ 's [5], the lack of those commutation relations introduces the necessity to enlarge the basis whenever we try to define a many-quon state, as we discuss later in this paper.

It is then our intention here to provide some basis to the use of the quon algebra as a tool to improve our approximated descriptions of many-body problems, and at the same time to pose some of the possible differences that arise with the introduction of that algebra, as compared to the more usually applied quantum algebras. In fact, the quonic Fock-like Hilbert space contains all possible symmetry permutations of the polynomials in the  $a^\dagger$ 's. As we shall see, even when we restrict ourselves to the symmetric representation, some important differences between both algebras can be noticed. Actually, a series of applications using a restricted quonic subspace was already done in the context of boson mappings[6],[7]. In section 3, we make those differences more explicit here, through the

solution of the three-dimensional harmonic (quon) oscillator as well as the analysis of the spectrum of a quonic version of the quantum rotor. Of course, these two examples can be viewed as a simple laboratory to test our main results in the construction of a many-body quon basis, as discussed in section 2. We believe that other interesting applications can be tackled in the future using our present results.

## 2 Many-Body Quon States

We start our discussion following the reasonings presented in reference[2] for the case of a two quons state. In that case we may write the following normalized states:

$$\frac{1}{\sqrt{1+q}}(a_1^\dagger)^2|0\rangle, \frac{1}{\sqrt{1+q}}(a_2^\dagger)^2|0\rangle, a_1^\dagger a_2^\dagger|0\rangle \text{ and } a_2^\dagger a_1^\dagger|0\rangle. \quad (1)$$

The last two states defined in (1) can be expanded in terms of a symmetric and a antisymmetric state in the form:

$$a_1^\dagger a_2^\dagger|0\rangle = \sqrt{\frac{1+q}{2}}|\phi_s\rangle + \sqrt{\frac{1-q}{2}}|\phi_a\rangle, \quad (2)$$

$$a_2^\dagger a_1^\dagger|0\rangle = \sqrt{\frac{1+q}{2}}|\phi_s\rangle - \sqrt{\frac{1-q}{2}}|\phi_a\rangle, \quad (3)$$

where,

$$|\phi_s\rangle = \frac{1}{\sqrt{2(1+q)}}(a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger)|0\rangle, \quad |\phi_a\rangle = \frac{1}{\sqrt{2(1-q)}}(a_1^\dagger a_2^\dagger - a_2^\dagger a_1^\dagger)|0\rangle$$

We may then conclude that the two-quon basis can be formed by one antisymmetric and three symmetric states. Also, once any observable must be represented by a symmetrical operator, the two states in equations (2) and (3) should be considered the same, in the sense that they give us the same observables. Another way to put this is to recognize that we can obtain the two-quon (orthonormal) basis by forming the overlap matrix from the states defined in (1) and diagonalize it. That procedure automatically lead us to the three symmetric and one antisymmetric states above. Then we can diagonalize any

observable within the symmetric and the antisymmetric subspaces separately. Of course, this important property can be readily generalized to any number of quons. For three quons for example, besides the well known symmetric and antisymmetric states, there are more exotic mixed symmetric states. To shorten the corresponding expressions we adopt the convention  $a_i^\dagger a_j^\dagger a_k^\dagger |0\rangle \equiv |ijk\rangle$ . Then we have for the (normalized) basis states in that case:

$$|S\rangle = \frac{1}{\sqrt{1+2q^2+2q+q^3}} \frac{1}{\sqrt{6}} [|ijk\rangle + |jik\rangle + |ikj\rangle + |jki\rangle + |kij\rangle + |kji\rangle] \quad (4)$$

$$|A\rangle = \frac{1}{\sqrt{1+2q^2-2q-q^3}} \frac{1}{\sqrt{6}} [|ijk\rangle - |jik\rangle - |ikj\rangle + |jki\rangle + |kij\rangle - |kji\rangle] \quad (5)$$

$$|MS1\rangle = \frac{1}{\sqrt{1-q^2+q-q^3}} \frac{1}{\sqrt{12}} [|ijk\rangle - |jik\rangle + 2|ikj\rangle + |jki\rangle - 2|kij\rangle - |kji\rangle] \quad (6)$$

$$|MS2\rangle = \frac{1}{\sqrt{1-q^2+q-q^3}} \frac{1}{2} [-|ijk\rangle - |jik\rangle + |jki\rangle + |kji\rangle] \quad (7)$$

$$|MS3\rangle = \frac{1}{\sqrt{1-q^2-q+q^3}} \frac{1}{2} [|ijk\rangle - |jik\rangle - |jki\rangle + |kji\rangle] \quad (8)$$

$$|MS4\rangle = \frac{1}{\sqrt{1-q^2-q+q^3}} \frac{1}{\sqrt{12}} [|ijk\rangle + |jik\rangle - 2|ikj\rangle + |jki\rangle - 2|kij\rangle + |kji\rangle] \quad (9)$$

where  $i, j, k = 1, 2, 3$ . Also, the cases  $i = j$ ,  $i = k$ ,  $j = k$  and  $i = j = k$  are automatically included in expressions (4) to (9), unless to a normalization factor which is  $q$ -independent. Evidently, the above basis states can be built from the well known procedure based in the Young tableaux method [8], or, as said before, through the diagonalization of the overlap matrix obtained from all possible order permutations from the state  $a_i^\dagger a_j^\dagger a_k^\dagger |0\rangle$ . In fact, the  $q$ -polynomials which appear in the square roots in equations (4) to (9), correspond to the eigenvalues of the overlap matrix and measure the degree of violation of statistics in the system. If we then choose  $q$  sufficiently close to 1(-1), we may restrict ourselves to the symmetric (antisymmetric) subspace, once the observables of the theory do not mix subspaces corresponding to different symmetries. At this respect it would be interesting to generalize our above expressions for the symmetric part of the

quonic space. This has a two-fold motivation: first of all many applications of the deformed algebras (for which only symmetric states are considered) to many-body problems are restricted to small deformations of the usual Lie algebra, i.e.,  $q$  close to 1[9] . We would like to compare some of those results with the equivalent solutions using the quon algebra. Secondly, the value of  $q$  very close ( but not equal) to 1 has the quite appealing idea to try to take in to account possible violations of bosonic statistics for systems in which the degrees of freedom are related to particles with a integer spin value but that are in fact composed by "fundamental " fermions.

It is then possible to show (see Appendix) that the most general symmetric state for a system of  $N$  quons can be written as:

$$|n_i n_j n_k \dots; S \rangle = \sqrt{\frac{n_i! n_j! n_k! \dots}{N! [N]!}} \hat{S}_N (a_i^\dagger)^{n_i} (a_j^\dagger)^{n_j} (a_k^\dagger)^{n_k} \dots |0 \rangle \quad (10)$$

where  $\hat{S}_N$  is a operator that generates all possible combinations that are symmetric under the permutation of any of the creation operators( as defined in the Appendix),  $n_i + n_j + n_k + \dots = N$  and [10]:

$$[N] = \frac{1 - q^N}{1 - q}, \quad (11)$$

with  $[N]! = [N][N-1]\dots[2][1]$  and  $[0]! = 1$ . Another important result that we are going to use next and which is also obtained in the Appendix, is the following:

$$a_i |n_i n_j n_k \dots; S \rangle = \sqrt{\frac{[N]}{N}} \sqrt{n_i} |n_i - 1, n_j n_k \dots; S \rangle \quad (12)$$

This last expression allows us to calculate matrix elements between symmetric quonic states with any number of quons. In the following section we discuss two simple examples and compare them to the deformed algebra results.

### 3 Applications and Results

In order to discuss some examples, it is important to recall that, according to reference[4], given a set of operators  $a_m, a_m^\dagger$  for which  $m = -j, \dots, +j$  , and such that

they obey the quon commutation relations, it can be proved that each  $a_m^\dagger$  behave as a  $\text{su}(2)$  irreducible tensor. In other words they obey the expected commutation relations with angular momentum operators, built from the corresponding number operators. However, those number operators present a complicated structure, and are written as an infinite series of the quonic creation and annihilation operators[1]. Also, it is not difficult to obtain a  $\text{su}(2)$  scalar from the quons. For example, we may define a quonic three dimensional harmonic oscillator by the Hamiltonian:

$$H_{osc}^q = \frac{\hbar\omega}{2} \{ (a_+^\dagger a_+ + a_-^\dagger a_- + a_0^\dagger a_0)(1+q) + 3 \} \quad (13)$$

where

$$a_+ = \frac{1}{\sqrt{2}}(a_1 + ia_2) , \quad a_- = \frac{1}{\sqrt{2}}(a_1 - ia_2) , \quad a_0 = a_3 . \quad (14)$$

First of all we note that, if  $a_1, a_2$  and  $a_3$  obey quon commutation relations, so does the set  $a_+, a_-$  and  $a_0$ . Secondly, the factor  $(1+q)$  comes from the fact that the Hamiltonian must be symmetrized and finally we can easily recover the usual harmonic oscillator Hamilton operator by simply choosing  $q = 1$ , from the above expression. In order to get the corresponding spectrum we should now diagonalize (13) inside each subspace formed by the states of a given symmetry of the whole permutation symmetry group and for a given number of quanta  $N = n_+ + n_- + n_0$ . Alternatively we could proceed with the diagonalization from the basis formed by all order permutations obtained from the state  $(a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} (a_0^\dagger)^{n_0} |0\rangle$ . This last procedure amounts however to a diagonalization in a non-orthonormal basis. On the other hand, the prior diagonalization of the overlap matrix, as done in the previous section for the  $N = 3$  case, corresponds to a partial diagonalization of the Hamiltonian, which then becomes block diagonal, each block corresponding to a given permutation symmetry.

As we said before, it is our intention here to make some comparisons to the deformed algebra results. Then we content ourselves with only the symmetric part of the solution, which is justifiable once we keep  $q$  close enough to 1. In that case, using equation (12) and its Hermitian conjugate, we readily find for the eigenvalues of our quonic harmonic oscillator, the remarkable simple result:

$$E_{osc}^q = \frac{\hbar\omega}{2} \{ [N](1+q) + 3 \} \quad (15)$$

Our quonic harmonic oscillator give us then a spectrum which is not equally spaced

but still depends on just one quantum number, the total number  $N$ . This is not the case in quantum algebras (see equation (28.38) in reference [9]) , for which the spectrum depends on  $N$  and on a additional quantum number  $l$ , related to the  $\text{su}_q(2)$  angular momentum. To get an idea of the effect of the deformation in the spectrum, we present in figure I a comparison to the regular oscillator for some selected values of the parameter  $q$ .

To better spot the angular momentum structure within the quon algebra we consider now the quantum rotor. A natural choice in this case is to follow a Schwinger type of prescription[11] for the definition of the angular momentum components:

$$L_+ = a_+^\dagger a_-, \quad L_- = a_-^\dagger a_+, \quad L_0 = \frac{1}{2} \{a_+^\dagger a_+ - a_-^\dagger a_-\} \quad (16)$$

We again restrict our results to the symmetric quon subspace. Using once more equation (12), we get the results:

$$\langle n'_+, n'_-; S | [L_+, L_-] | n_+, n_-; S \rangle = \frac{[N]}{N} (n_+ - n_-) \delta_{n'_+ n_+} \delta_{n'_- n_-} \quad (17)$$

and,

$$\langle n'_+, n'_-; S | 2L_0 | n_+, n_-; S \rangle = \frac{[N]}{N} (n_+ - n_-) \delta_{n'_+ n_+} \delta_{n'_- n_-} \quad (18)$$

where now  $N = n_+ + n_-$ . Using the same type of calculation we may also prove that:

$$\langle n'_+, n'_-; S | [L_0, L_+] | n_+, n_-; S \rangle = \langle n'_+, n'_-; S | L_+ | n_+, n_-; S \rangle \quad (19)$$

$$\langle n'_+, n'_-; S | [L_0, L_-] | n_+, n_-; S \rangle = - \langle n'_+, n'_-; S | L_- | n_+, n_-; S \rangle \quad (20)$$

The above results show that the operators defined in (16), behave as genuine angular momentum components within the symmetric subspace. All we need now is to obtain the matrix element of the operator  $L^2$ , which gives:

$$\langle n'_+, n'_-; S | L^2 | n_+, n_-; S \rangle = \frac{[N]}{2} \left( \frac{[N]}{2} + 1 \right) \delta_{n'_+ n_+} \delta_{n'_- n_-} \quad (21)$$

Assuming the correspondence  $n_+ = l + m$  and  $n_- = l - m$  we finally obtain for our  $q$ -rotor spectrum:

$$E_l^q = A \frac{[2l]}{2} \left\{ \frac{[2l]}{2} + 1 \right\} \quad (22)$$

with  $A$  being the inertia constant. Again, we see that, although we get the right limit for  $q=1$ , the spectrum given by equation (22) is different from the one obtained through quantum algebra techniques (see equation (19.3) in [9]). At this respect, one interesting result that emerges from the deformed algebra rotor, is its ability to describe stretching effects as experimentally observed in the spectra of heavy nuclei and molecules, with the introduction of a single parameter[12]. In our case, we could test the applicability of our previous results doing the same sort of analysis, using expression (22). In figure II we show the experimental spectrum of the fundamental rotational band in the  $^{240}\text{Pu}$  nucleus, chosen here as typical sample, together with the one obtained from our quonic rotor. We choose a  $q$ -value that minimizes the differences between the theoretical spectrum and the experimental one, within the interval allowed by the quon algebra. It is interesting to observe that a  $q$ -value slightly smaller than 1 is enough to produce the desired modifications in order to take in to account the stretching effects just mentioned, as we can observe particularly for higher values of the angular momentum, compared to the rigid rotor result. Also shown in the same figure is the best fit spectrum obtained using the quantum algebra approach from reference[12].

## 4 Conclusions

In this letter we have discussed and presented a method to build in a systematic way, a basis of states that represent a system of identical quons. The novel feature of that type of states lie in the fact that, once quons obey commutation relations which interpolate between bosons and fermions, all kind of permutation symmetries can be accommodated in a many-quon state. The probability that each type of symmetry occur is then controlled by a single parameter  $q$ . Once any observable should be symmetric by any particle exchange, a classification of the states by their permutational symmetry amounts to a partial diagonalization of the corresponding operator in the whole quonic space. In order to make some contact with the deformed algebra results, we have kept our attention to the totally symmetric subspace, for which we could find a closed general expression for a state with any number of quons, as well as for the action of an operator on it. Two



simple examples were then considered here within that point of view: a quonic version of the three-dimensional harmonic oscillator and a rotor model based on the quon algebra. As a by product we have found out the interesting result that the angular momentum operator written in terms of quons and within the symmetric subspace behaves as usual  $\text{su}(2)$  angular momentum operators, having the same functional form as in the case of regular bosons. Also, a comparison of both examples with the results previously obtained within quantum algebras, show a very distinct energy spectrum distribution in the case of the harmonic oscillator where the same degree of degeneracy observed in the usual bosonic oscillator is recovered, contrary to what happens when we use the quantum  $\text{su}_q(2)$  algebra. As for the rigid rotor, though we have found an energy spectrum very close in both cases, the angular momentum operator properties are quite different from the  $\text{su}_q(2)$  properties.

Quantum or q-deformed algebras are by now considered a very powerful tool to deal with physical systems for which usual algebras can not take in to account some of their properties. The quon algebra, which is considered in the literature as a particular case[10] of those algebras, present subtle differences that may reflect some important and even fundamental differences in what concerns the interpretation of the final results. As mentioned in the Introduction, an interesting point that deserves some investigation in the near future is the analysis of bosonic systems that are in fact composed by fermions, once some observed deviations from a true boson behavior could in principle be realized by the quon algebra in a natural way.

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### Appendix

In this appendix we prove two important results, eqs.(10,12), given in the main text. We begin with the most general symmetric (not normalized) N quons state in an arbitrary basis:

$$\hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k}...|0> \equiv \frac{1}{n_i!n_j!n_k!...} \sum_{P_N} a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger ... a_{\alpha_{n_i}}^\dagger a_{\alpha_{n_i+1}}^\dagger ... a_{\alpha_{n_i+n_j+1}}^\dagger ... a_{\alpha_N}^\dagger |0> \quad , \quad (23)$$

where  $N=n_i + n_j + n_k + \dots$  and the summation runs over all the  $N!$  permutations,  $P_N$ , in the indices  $\alpha_1, \alpha_2, \dots, \alpha_N$ . We order these indices such that  $\alpha_1, \alpha_2, \dots, \alpha_{n_i}$  corresponds to the i-state,  $\alpha_{n_i+1}, \alpha_{n_i+2}, \dots, \alpha_{n_i+n_j}$  to the j-state and so on. The factorials under the denominator accounts for repeated terms in the summation and eq.(23) is the definition of the  $\hat{S}_N$  operator. We now prove by induction the following result:

$$a_i \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle = [N] \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle, \quad (24)$$

where  $[N]$  is given in eq.(11). It is easy to show that the relation above is valid for  $N=1$  ( $\hat{S}_0 = I$ , the identity operator). We assume that it is also true for a  $N-1$  quons state, i. e.,

$$a_i \hat{S}_{N-1}(a_i^\dagger)^{n'_i}(a_j^\dagger)^{n'_j}(a_k^\dagger)^{n'_k} \dots |0\rangle = [N-1] \hat{S}_{N-2}(a_i^\dagger)^{n'_i-1}(a_j^\dagger)^{n'_j}(a_k^\dagger)^{n'_k} \dots |0\rangle, \quad (25)$$

where  $N-1=n'_i+n'_j+n'_k+\dots$ . One can shows the following property of the symmetrization operator:

$$\begin{aligned} \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle &= a_i^\dagger \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle \\ &+ a_j^\dagger \hat{S}_{N-1}(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j-1}(a_k^\dagger)^{n_k} \dots |0\rangle + a_k^\dagger \hat{S}_{N-1}(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k-1} \dots |0\rangle + \dots \end{aligned} \quad (26)$$

This property follows from the definition of the symmetrization operator, eq.(23), and its rearrangement as given below:

$$\begin{aligned} \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle &= \frac{1}{n_i!n_j!n_k! \dots} \\ &\cdot (a_{\alpha_1}^\dagger \sum_{P_{N-1}} \hat{a}_{\alpha_1}^\dagger a_{\alpha_2}^\dagger \dots a_{\alpha_N}^\dagger |0\rangle + a_{\alpha_2}^\dagger \sum_{P_{N-1}} a_{\alpha_1}^\dagger \hat{a}_{\alpha_2}^\dagger a_{\alpha_3}^\dagger \dots a_{\alpha_N}^\dagger |0\rangle \\ &+ \dots + a_{\alpha_{n_i}}^\dagger \sum_{P_{N-1}} a_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_{n_i}}^\dagger a_{\alpha_{n_i+1}}^\dagger \dots a_{\alpha_N}^\dagger |0\rangle + \dots + a_{\alpha_N}^\dagger \sum_{P_{N-1}} a_{\alpha_1}^\dagger \dots a_{\alpha_{N-1}}^\dagger \hat{a}_{\alpha_N}^\dagger |0\rangle) \end{aligned} \quad (27)$$

where the hat symbol on the creation operator means that it is omitted in that position. The first  $n_i$  terms above are equal, since  $\alpha_1, \alpha_2, \dots, \alpha_{n_i}$  are associated to the i-state, the same argument may be used for the next  $n_j$  terms and so on. So the property given in eq.(26) is proved.

From eq.(26) and the q-mutation relation, we obtain for the action of the annihilation operator,  $a_i$ , in the  $N$  quons symmetric state the result as follows:

$$a_i \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle = \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0\rangle$$

$$\begin{aligned}
& + q( a_i^\dagger a_i \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > + a_j^\dagger a_i \hat{S}_{N-1}(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j-1}(a_k^\dagger)^{n_k} \dots |0 > \\
& + a_k^\dagger a_i \hat{S}_{N-1}(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k-1} \dots |0 > )
\end{aligned} \tag{28}$$

We now use eq.(25) to rewrite the term between parenthesis and get

$$\begin{aligned}
& a_i \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > = \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > \\
& + q[N-1]( a_i^\dagger \hat{S}_{N-2}(a_i^\dagger)^{n_i-2}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > + a_j^\dagger \hat{S}_{N-2}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j-1}(a_k^\dagger)^{n_k} \dots |0 > \\
& + a_k^\dagger \hat{S}_{N-2}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k-1} \dots |0 > )
\end{aligned} \tag{29}$$

Finally using eq.(26) in the term between parenthesis and the q-number property,  $[N]=1+q[N-1]$ , we may rearrange the above expression as:

$$a_i \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > = [N] \hat{S}_{N-1}(a_i^\dagger)^{n_i-1}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > . \tag{30}$$

and the proof by induction is finished.

Now we obtain the normalized symmetric state. Let us write:

$$|n_i, n_j, n_k, \dots; S > \equiv A_{n_i n_j n_k \dots} \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > . \tag{31}$$

So the normalization constant,  $A_{n_i n_j n_k \dots}$  is determined by:

$$\begin{aligned}
1 & = \langle n_i, n_j, n_k, \dots; S | n_i, n_j, n_k, \dots; S \rangle = A_{n_i n_j n_k \dots}^2 \frac{N!}{n_i! n_j! n_k!} \\
& \cdot \langle 0 | \dots (a_k)^{n_k} (a_j)^{n_j} (a_i)^{n_i} \hat{S}_N(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}(a_k^\dagger)^{n_k} \dots |0 > = A_{n_i n_j n_k \dots}^2 \frac{N! [N]!}{n_i! n_j! n_k!} ,
\end{aligned}$$

where in order to get the result above, eq.(30), was iterated. So the normalization factor in eq.(10) is obtained. From eq.(30) and the normalization factor obtained above, eq.(12) follows trivially.

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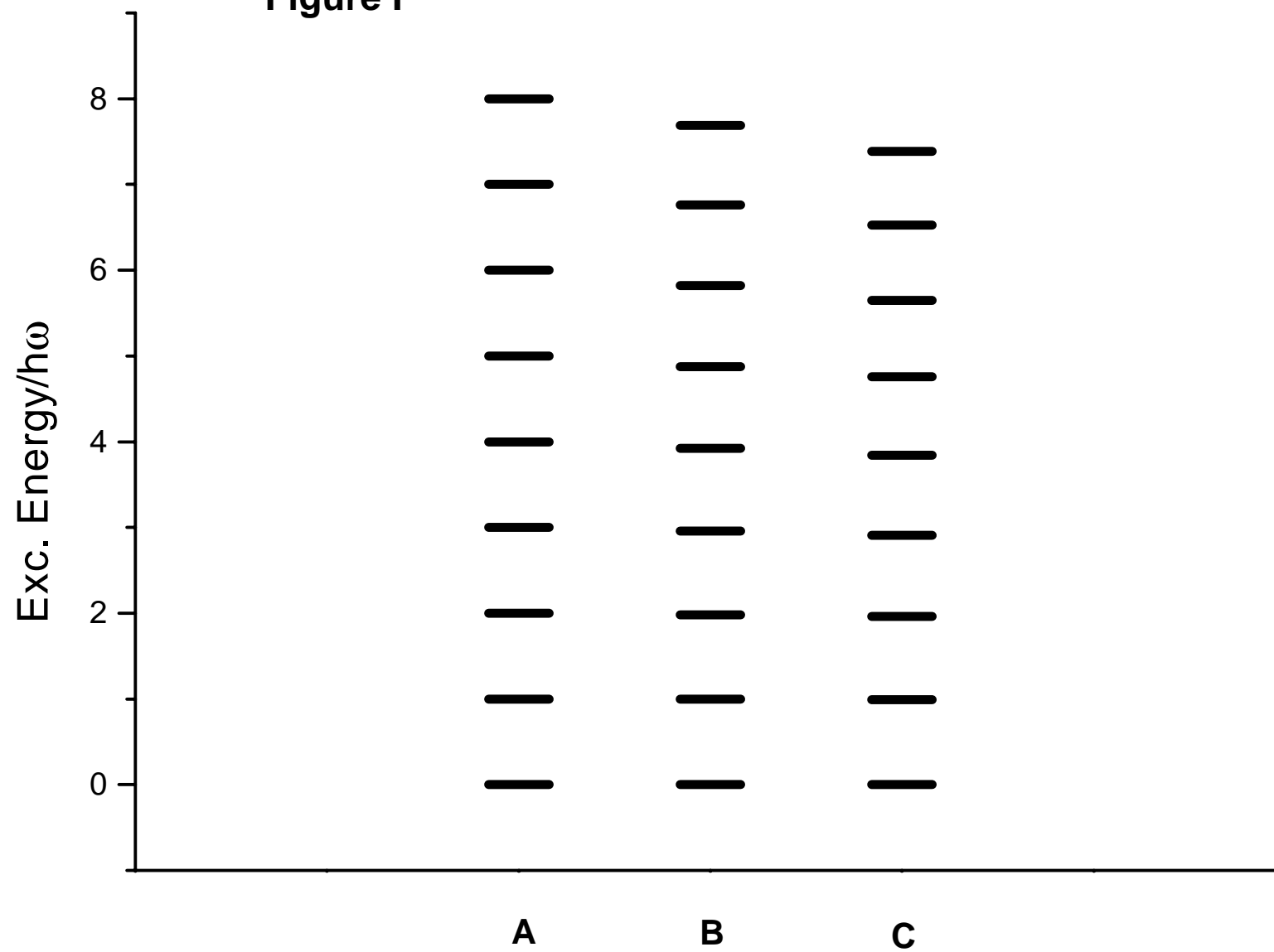
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## FIGURE CAPTIONS

Figure I. Harmonic oscillator spectrum (**A**) compared to the quonic harmonic oscillator spectra obtained for two different values of the deformation parameter :  $q = 0.99$  (**B**) and  $q = 0.98$  (**C**).

Figure II. The experimental (**A**) spectrum for the  $^{244}\text{Pu}$  fundamental rotational band compared to the quonic rotor result (**B**), deformed algebra result from reference [12] (**C**) and the usual quantum rigid rotor result (**D**). Spectrum **B** was obtained using  $q = 0.99478$ .

**Figure I**



**Figure II**

